

Due Fri: 6.2 - Angle and Orthogonality in Inner Product Spaces

nonempty set of objects → 10 axioms + scalar mult → vector space → defined inner product → 4 axioms

**Definition:** The angle  $\theta$  between vectors  $u$  and  $v$  in a real inner product space  $V$  is  $\theta = \cos^{-1} \left( \frac{\langle u, v \rangle}{\|u\| \|v\|} \right)$ .  $\frac{|\langle u, v \rangle|}{\|u\| \|v\|} \leq 1$  (As we saw with the dot product)

#5 Find the cosine of the angle between  $A$  and  $B$  with respect to the standard inner product on  $M_{22}$ .

$A = \begin{bmatrix} 2 & 6 \\ 1 & -3 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$        $\langle A, B \rangle = \text{tr}(A^T B) = 2(3) + 6(2) + 1(1) - 3(0) = 19$

$\|A\| = \sqrt{\langle A, A \rangle} = \sqrt{2^2 + 6^2 + 1^2 + (-3)^2} = \sqrt{50} = 5\sqrt{2}$

$\|B\| = \sqrt{3^2 + 2^2 + 1^2 + 0^2} = \sqrt{14}$

$\cos \theta = \frac{19}{10\sqrt{7}} \approx 0.72$        $\theta \approx 44^\circ$

**Theorem 6.2.1** Cauchy-Schwarz Inequality (generalization of Theorem 3.2.4)  
 If  $u$  and  $v$  are vectors in a real inner product space  $V$ , then  $|\langle u, v \rangle| \leq \|u\| \|v\|$ .

A necessary result in order to consider the angle between two vectors in an inner product space.

**Theorem 6.2.2** Triangle inequalities (generalization of Theorem 3.2.5)  
 If  $u, v$ , and  $w$  are vectors in a real inner product space  $V$ , then:

- a.  $\|u + v\| \leq \|u\| + \|v\|$  (triangle inequality for vectors)
- b.  $d(u, v) \leq d(u, w) + d(w, v)$  (triangle inequality for distances)

pf: a)  $\|\vec{u} + \vec{v}\|^2 = \langle \vec{u} + \vec{v}, \vec{u} + \vec{v} \rangle$   
 $= \langle \vec{u}, \vec{u} \rangle + 2\langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{v} \rangle$   
 $\leq \langle \vec{u}, \vec{u} \rangle + 2|\langle \vec{u}, \vec{v} \rangle| + \langle \vec{v}, \vec{v} \rangle$  (prop. of abs. value)

$$\leq \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2 \quad \text{Cauchy-Schwarz inequality}$$

$$= (\|\vec{u}\| + \|\vec{v}\|)^2$$

Taking square roots finishes the proof.

$$b) d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \|(\vec{u} - \vec{w}) + (\vec{w} - \vec{v})\| \quad \begin{array}{l} \|\vec{a} + \vec{b}\| \\ \|\vec{a}\| + \|\vec{b}\| \end{array}$$

$$\leq \|\vec{u} - \vec{w}\| + \|\vec{w} - \vec{v}\|$$

$$= d(\vec{u}, \vec{w}) + d(\vec{w}, \vec{v}) \quad \checkmark$$

**Definition:** Two vectors  $\mathbf{u}$  and  $\mathbf{v}$  in a real inner product space  $V$  are **orthogonal** if  $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ .

**#10** Show that the vectors are orthogonal with respect to the standard inner product on  $P_2$ .

$$\mathbf{p} = 2 - 3x + x^2, \mathbf{q} = 4 + 2x - 2x^2$$

$$\langle \vec{p}, \vec{q} \rangle = 2(4) - 3(2) + 1(-2) = 0 \quad \checkmark$$

**#33** Let  $C[-1, 1]$  have the integral inner product  $\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^1 p(x)q(x)dx$  and let  $\mathbf{p} = p(x) = x^2 - x$  and  $\mathbf{q} = q(x) = x + 1$ .

a. Find  $\langle \mathbf{p}, \mathbf{q} \rangle$ .

b. Find  $\|\mathbf{p}\|$  and  $\|\mathbf{q}\|$ .

$$a. \langle \vec{p}, \vec{q} \rangle = \int_{-1}^1 (x^3 + x^2 - x^2 - x)dx = \int_{-1}^1 (x^3 - x)dx = 0$$

*Symmetric interval*  $\downarrow$  *odd function*

$$\vec{p} \perp \vec{q}$$

$$b. \|\vec{p}\| = \sqrt{\langle \vec{p}, \vec{p} \rangle} = \left[ \int_{-1}^1 (x^2 - x)^2 dx \right]^{1/2} = \left[ \int_{-1}^1 (x^4 - 2x^3 + x^2) dx \right]^{1/2}$$

$$= \sqrt{2 \left( \frac{1}{5} + \frac{1}{3} \right)} = \sqrt{\frac{16}{15}} = \frac{4}{\sqrt{15}}$$

$$\|\vec{q}\| = \left[ \int_{-1}^1 (x^2 + 2x + 1) dx \right]^{1/2} = \sqrt{2 \left( \frac{1}{3} + 1 \right)} = \sqrt{\frac{8}{3}} = \frac{2\sqrt{2}}{\sqrt{3}}$$

Explore: What is  $\|\vec{p} + \vec{q}\|^2$ ? What about  $\|\vec{p}\|^2 + \|\vec{q}\|^2$ ?

$$\|\vec{p} + \vec{q}\|^2 = \langle \vec{p} + \vec{q}, \vec{p} + \vec{q} \rangle = \int_{-1}^1 (x^2 + 1)^2 dx$$

$$= 2 \int_{-1}^1 (x^4 + 2x^2 + 1) dx = 2 \left( \frac{1}{5} + \frac{2}{3} + 1 \right) = \frac{56}{15}$$

$$\|\vec{p}\|^2 + \|\vec{q}\|^2 = \frac{16}{15} + \frac{8}{3} = \frac{56}{15}$$

$$\text{Here, } \|\vec{p} + \vec{q}\|^2 = \|\vec{p}\|^2 + \|\vec{q}\|^2$$

### Theorem 6.2.3 Generalized Theorem of Pythagoras

If  $\vec{u}$  and  $\vec{v}$  are orthogonal vectors in a real inner product space, then

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

$$\text{pf: } \|\vec{u} + \vec{v}\|^2 = \langle \vec{u} + \vec{v}, \vec{u} + \vec{v} \rangle$$

$$= \|\vec{u}\|^2 + 2\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2$$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 \quad \checkmark$$

**Example:** Let  $p = p(x) = 1 - 2x^2$ ,  $q = q(x) = 4 - 2x + x^2$ ,  $r = r(x) = x + 2x^2$  and let  $P_2$  have the standard inner product.

- a. Compute the following:  $\langle p, q \rangle$  and  $\langle q, r \rangle$ . - different inner product spaces
- b. Now let  $P_2$  have the inner product  $\langle p, q \rangle = a_0b_0 + a_1b_1 + ka_2b_2$ . Find  $k$  so that  $p$  and  $q$  are orthogonal.
- c. Using this new inner product, compute  $\langle q, r \rangle$ .

a.  $\langle \vec{p}, \vec{q} \rangle = 1(4) - 2(1) = 2$   $\vec{p} \not\perp \vec{q}$   
 $\langle \vec{q}, \vec{r} \rangle = -2(1) + 1(2) = 0$   $\vec{q} \perp \vec{r}$

b.  $\langle \vec{p}, \vec{q} \rangle = 1(4) - k(2)(1) = 4 - 2k$   
 $\vec{p} \perp \vec{q} \Rightarrow 4 - 2k = 0 \Rightarrow k = 2$   $\vec{p} \perp \vec{q}$

c.  $\langle \vec{q}, \vec{r} \rangle = -2(1) + 2(2)(1) = 2$   $\vec{q} \not\perp \vec{r}$

Orthogonality depends on the vectors and the inner product being used.

#17 Do there exist scalars  $k$  and  $l$  such that the vectors  $p_1 = 2 + kx + 6x^2$ ,  $p_2 = l + 5x + 3x^2$ ,  $p_3 = 1 + 2x + 3x^2$  are mutually orthogonal with respect to the standard inner product on  $P_2$ ? all pairs are orthogonal

$\langle \vec{p}_1, \vec{p}_2 \rangle = 2l + 5k + 18$ . If  $\vec{p}_1 \perp \vec{p}_2$ ,  $2l + 5k + 18 = 0$   
 $\langle \vec{p}_2, \vec{p}_3 \rangle = l + 10 + 9$ .  $\vec{p}_2 \perp \vec{p}_3 \Rightarrow l = -19$ .  
 $\langle \vec{p}_1, \vec{p}_3 \rangle = 2 + 2k + 18$ .  $\vec{p}_1 \perp \vec{p}_3 \Rightarrow 2k = -20 \Rightarrow k = -10$

$\langle \vec{p}_1, \vec{p}_2 \rangle = 2(-19) + 5(-10) + 18 \neq 0$

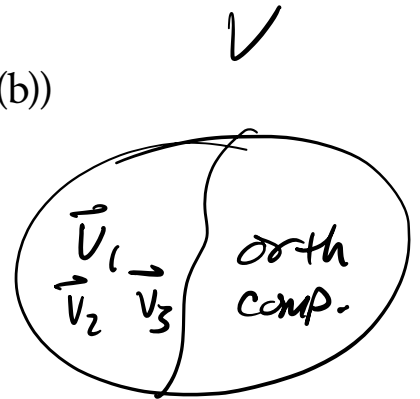
No  $k$  &  $l$  work

**Definition:** (inner product space analog of Definition 2 in Section 4.9) If  $W$  is a subspace of a real inner product space  $V$ , then the set of all vectors in  $V$  that are orthogonal to every vector in  $W$  is called the **orthogonal complement** of  $W$  and is denoted by the symbol  $W^\perp$  (pronounced “ $W$  perp”).

**Theorem 6.2.4** (generalization of Theorem 4.9.6 (a) and (b))

If  $W$  is a subspace of a real inner product space  $V$ , then:

- a)  $W^\perp$  is a subspace of  $V$ .
- b)  $W \cap W^\perp = \{0\}$ .



**Theorem 6.2.5** (generalization of Theorem 4.9.6(c))

If  $W$  is a subspace of a real finite-dimensional inner product space  $V$ , then the orthogonal complement of  $W^\perp$  is  $W$ , that is,  $(W^\perp)^\perp = W$ .

#27 Find a basis for the orthogonal complement of the subspace of  $(\mathbb{R}^4)$  spanned by the vectors  $\mathbf{v}_1 = (1, 4, 5, 2)$ ,  $\mathbf{v}_2 = (2, 1, 3, 0)$ ,  $\mathbf{v}_3 = (-1, 3, 2, 2)$ .

*dot product*

$$\begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 5 & 3 & 2 \\ 2 & 0 & 2 \end{bmatrix} \vec{x} = \vec{0} \quad \vec{x} \in \mathbb{R}^3$$

~~wrong space~~

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{0} \quad \vec{x} \in \mathbb{R}^4$$

↳ generalized:  $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = 0$

The solution space is the orthogonal complement to the row space.

$$\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 7 & -7 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & -2/7 \\ 0 & 1 & 1 & 4/7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -x_3 + \frac{2}{7}x_4, \quad x_2 = -x_3 - \frac{4}{7}x_4$$

Let  $s = x_3$  and  $t = x_4$ .

$$\vec{x} = \begin{bmatrix} -s + \frac{2}{7}t \\ -s - \frac{4}{7}t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix} t$$

A basis for the orthogonal complement is  $\{(1, 1, -1, 0), (2, -4, 0, 7)\}$ .

$$\vec{v}_1 = (1, 4, 5, 2), \quad \vec{v}_2 = (2, 1, 3, 0), \quad \vec{v}_3 = (-1, 3, 2, 2)$$

Pf of Cauchy-Schwarz inequality:

That is, pf that for  $\vec{u}, \vec{v} \in V$  an inner product space,  $|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \|\vec{v}\|$

If  $\vec{u} = \vec{0}$ , then  $\langle \vec{u}, \vec{v} \rangle = 0$  and  $\|\vec{u}\| = 0$ .

If  $\vec{u} \neq \vec{0}$ , then let  $\langle \vec{u}, \vec{u} \rangle = a$ ,

$2\langle \vec{u}, \vec{v} \rangle = b$ ,  $\langle \vec{v}, \vec{v} \rangle = c$ , and let  $t$  be any real number.

$$0 \leq \langle t\vec{u} + \vec{v}, t\vec{u} + \vec{v} \rangle = \langle \vec{u}, \vec{u} \rangle t^2 + 2\langle \vec{u}, \vec{v} \rangle t + \langle \vec{v}, \vec{v} \rangle$$

$$at^2 + bt + c \geq 0$$

Then  $b^2 - 4ac \leq 0$

$$4\langle \vec{u}, \vec{v} \rangle^2 - 4\|\vec{u}\|^2 \|\vec{v}\|^2 \leq 0$$

$$\Rightarrow \langle \vec{u}, \vec{v} \rangle^2 \leq \langle \vec{u}, \vec{u} \rangle \langle \vec{v}, \vec{v} \rangle \quad \text{one form}$$

$$\text{OR } \langle \vec{u}, \vec{v} \rangle^2 \leq \|\vec{u}\|^2 \|\vec{v}\|^2 \quad \text{another form}$$

$$\text{OR } |\langle \vec{u}, \vec{v} \rangle| = \|\vec{u}\| \|\vec{v}\| \quad \text{yet another form}$$

